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Harmonic approximation and Sarason's-type theorem

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Abstract

In this paper uniform approximation of bounded harmonic functions on an arbitrary open set in Euclidean space by harmonic functions arising as solutions of the classical or generalized Dirichlet problem is studied. In particular, an analogue of Sarason's $H^\infty + C$ theorem (known from the theory of algebras of analytic functions) is established for harmonic functions.

1. Main results

In the present paper, we study uniform approximation of bounded harmonic functions on an arbitrary open bounded subset U of the Euclidean space $\mathbb{R}^d$, $d \geqslant 2$, by means of harmonic functions arising as solutions of the classical or generalized Dirichlet problem. As a consequence of our results we establish an analogue of Sarason's $H^\infty + C$ theorem (known from the theory of algebras of analytic functions, see, e.g. [5]) for harmonic functions.

As usual, let $\mathscr{C}(\bar{U})$ and $\mathscr{C}(\partial U)$ be the Banach space of continuous real functions on the closure $\bar{U}$ and the boundary ∂U of U, respectively. The closed subspace of $\mathscr{C}(\bar{U})$ consisting of all functions $h \in \mathscr{C}(\bar{U})$ which are harmonic on U is denoted by $H(U)$, and $\mathscr{H}_b(U)$ stands for the space of bounded harmonic functions on U. Given $f \in \mathscr{C}(\partial U)$, we write $H_U f$ for the Perron–Wiener–Brelot solution of the Dirichlet problem on U for the boundary condition f.

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For every bounded function g on U, we define functions g_*, g^* on $\bar{U}$ by

$$g_*(x) = \liminf_{y\in U, y\to x} g(y), \quad g^*(x) = \limsup_{y\in U, y\to x} g(y)$$

so that

$$\operatorname{osc}(g)(x) := g^*(x) - g_*(x)$$

is the oscillation of g at $x\in\bar{U}$. We note that of course g_* is l.s.c., g^* is u.s.c., and $g_* \leqslant g^*$ whence $\operatorname{osc}(g)$ is a positive u.s.c. function.

Our first approximation result, which is of interest for non-regular domains only, reads as follows:

Proposition 1. *Let $\varphi\in\mathscr{C}(\partial U)$ and $f_0\in\mathscr{C}(\bar{U})$ such that $f_0>0$ and $\operatorname{osc}(H_U\varphi)<f_0$ on ∂U. Then there exists a function $h\in H(U)$ such that*

$$|h - H_U\varphi| < f_0 \quad \text{on } U. \tag{1}$$

This shows again that, for every $\varphi\in\mathscr{C}(\partial U)$, there exists a sequence (h_n) in $H(U)$ which converges locally uniformly to $H_U\varphi$ on U, a result already obtained in [3].

Given φ and f_0 such that $\operatorname{osc}(H_U\varphi)<f_0$, it is of course possible to choose $\alpha<1$ such that $\operatorname{osc}(H_U\varphi)<\alpha f_0$ and consequently $|h-H_U\varphi|<\alpha f_0$ on U. However, it may be impossible to replace f_0 in (1) by some αf_0 with a constant $\alpha<1$ which does not depend on the choice of φ, f_0.

Proposition 2. *There exists a bounded domain U in $\mathbb{R}^d$, $d\geqslant 3$, having the following property: For every $0<\alpha<1$, there exists $\varphi\in\mathscr{C}(\partial U)$ such that $\operatorname{osc}(H_U\varphi)\leqslant 1$ and there is no function $h\in H(U)$ satisfying*

$$|h - H_U\varphi| < \alpha \quad \text{on } U.$$

The next result shows to what extent functions in $\mathscr{H}_{\mathrm{b}}(U)$ can be approximated by functions in $H(U)$.

Theorem 3. *Let $g\in\mathscr{H}_{\mathrm{b}}(U)$ and $h_0\in H(U)$ such that $\operatorname{osc}(g)<h_0$ and let K be a compact subset of U. Then there exists a function $h\in H(U)$ satisfying*

$$|g-h| < \tfrac{3}{2}h_0 \quad \text{on } U \quad \text{and} \quad |g-h| < \tfrac{1}{2}h_0 \quad \text{on } K.$$

More precisely, for every $f\in\mathscr{C}(\bar{U})$ such that $0\leqslant f\leqslant h_0$ on $\bar{U}$ and $f=h_0$ on ∂U, there exists a function $h\in H(U)$ satisfying

$$|g-h| < \tfrac{1}{2}h_0 + f \quad \text{on } U.$$

Using the set U constructed in the proof of Proposition 2 we are able to show that such an approximation cannot be improved.

An application of Theorem 3 leads to the following representation of functions in $\mathscr{H}_{\mathrm{b}}(U) + \mathscr{C}(\bar{U})|_U$:

Proposition 4. *For every function* $F \in \mathscr{H}_{\mathrm{b}}(U) + \mathscr{C}(\bar{U})|_U$, *there exist* $g \in \mathscr{H}_{\mathrm{b}}(U)$ *and* $f \in \mathscr{C}(\bar{U})$ *such that* $F = g + f$ *on* U *and* $\sup |g|(U) \leqslant 4 \sup |F|(U)$.

Then it is easy to obtain the following Sarason's-type theorem:

Corollary 5. $\mathscr{H}_{\mathrm{b}}(U) + \mathscr{C}(\bar{U})|_U$ *is closed with respect to uniform convergence.*

For the case of a regular set U, this is proved in [4].

2. Proofs

Let us start with the following observation:

Lemma 6. *Let* $g \in \mathscr{H}_{\mathrm{b}}(U)$ *and* $f_0 \in \mathscr{C}(\partial U)$ *such that* $\mathrm{osc}(g) \leqslant f_0$ *on* ∂U. *Then there exists* $\varphi \in \mathscr{C}(\partial U)$ *such that* $|H_U \varphi - g| \leqslant \frac{1}{2} H_U f_0$.

Proof. Since $g^* - \frac{1}{2} f_0 \leqslant g_* + \frac{1}{2} f_0$ on ∂U and $g^* - \frac{1}{2} f_0$ ($g_* + \frac{1}{2} f_0$ resp.) is u.s.c. on ∂U (l.s.c. on ∂U resp.), there exists $\varphi \in \mathscr{C}(\partial U)$ satisfying

$$g^* - \tfrac{1}{2} f_0 \leqslant \varphi \leqslant g_* + \tfrac{1}{2} f_0 \quad \text{on } \partial U.$$

By definition of solutions to the generalized Dirichlet problem

$$g \leqslant H_U(\varphi + \tfrac{1}{2} f_0) \quad \text{and} \quad H_U(\varphi - \tfrac{1}{2} f_0) \leqslant g.$$

By linearity of H_U we conclude that $|H_U \varphi - g| \leqslant \frac{1}{2} H_U f_0$. □

Obviously, Proposition 1 follows immediately taking $g = H_U \varphi$ and $f_1 = 0$ in the following lemma.

Lemma 7. *Let* $\varphi \in \mathscr{C}(\partial U)$, $g \in \mathscr{H}_{\mathrm{b}}(U)$, *and* $f_0, f_1 \in \mathscr{C}(\bar{U})$ *such that*

$$\mathrm{osc}(g) < f_0 \quad \text{on } \bar{U} \quad \text{and} \quad g - \tfrac{1}{2} f_1 \leqslant H_U \varphi \leqslant g + \tfrac{1}{2} f_1 \quad \text{on } U.$$

Then there exists $h \in H(U)$ *such that*

$$|g - h| < f_0 + \tfrac{1}{2} f_1 \quad \text{on } U.$$

Proof. By Bliedtner and Hansen [1, VI.8.5], there exists a bounded sequence (h_n) in $H(U)$ such that, for every $x \in \bar{U}$,

$$\lim_{n \to \infty} h_n(x) = \varepsilon_x^{U^c}(\varphi).$$

Hence, for every $y \in U$,

$$\lim_{n \to \infty} h_n(y) = H_U \varphi(y).$$

Fix $z \in \partial U$. Then there exists a sequence (y_m) in U such that $\lim_{m \to \infty} y_m = z$ and the harmonic measures $\varepsilon_{y_m}^{U^c}$ converge vaguely to $\varepsilon_z^{U^c}$ as m tends to infinity (see [1, VI.2.6]).

In particular,

$$\lim_{m\to\infty} H_U\varphi(y_m) = \lim_{m\to\infty} \varepsilon_{y_m}^{U^c}(\varphi) = \varepsilon_z^{U^c}(\varphi) = \lim_{n\to\infty} h_n(z)$$

whence

$$g_*(z) - \tfrac{1}{2}f_1(z) \leqslant \lim_{m\to\infty} H_U\varphi(y_m) = \lim_{n\to\infty} h_n(z) \leqslant g^*(z) + \tfrac{1}{2}f_1(z).$$

So

$$g_* - \tfrac{1}{2}f_1 \leqslant \lim_{n\to\infty} h_n \leqslant g^* + \tfrac{1}{2}f_1 \quad \text{on } \bar{U}.$$

By assumption, $g^* - g_* < f_0$. Define

$$v := -g + f_0 + \tfrac{1}{2}f_1.$$

Then $v_* = -g^* + f_0 + \frac{1}{2}f_1 > -g_* + \frac{1}{2}f_1$ and therefore

$$v_* + \lim_{n\to\infty} h_n > 0 \quad \text{on } \bar{U}. \tag{2}$$

The functions $v_* + h_n$ are l.s.c. and uniformly bounded. Given $k \in \mathbb{N}$, let

$$\mathscr{F}_k := \operatorname{conv}\{h_n : n \in \mathbb{N}, n \geqslant k\}.$$

For the moment, fix $k \in \mathbb{N}$ and let $\mu \neq 0$ be a measure on $\bar{U}$. By Lebesgue's dominated convergence theorem, (2) implies that

$$\lim_{n\to\infty} \int (v_* + h_n)\, d\mu = \int \left(v_* + \lim_{n\to\infty} h_n\right) d\mu > 0.$$

Therefore there exists $u_k \in \mathscr{F}_k$ such that

$$v_* + u_k > 0 \quad \text{on } \bar{U} \tag{3}$$

(see [2] or [1, I.1.9]). Obviously,

$$\lim_{k\to\infty} u_k = \lim_{n\to\infty} h_n. \tag{4}$$

Next let

$$w := g + f_0 + \tfrac{1}{2}f_1.$$

Then $w_* = g_* + f_0 + \frac{1}{2}f_1 > g^* + \frac{1}{2}f_1$ and therefore

$$w_* - \lim_{k\to\infty} u_k = w_* - \lim_{n\to\infty} h_n > 0 \quad \text{on } \bar{U}. \tag{5}$$

Arguing as before we obtain a (finite) convex combination h of the functions u_k such that

$$w_* - h > 0 \quad \text{on } \bar{U}.$$

Of course, (3) implies that

$$v_* + h > 0 \quad \text{on } \bar{U}$$

as well. In particular, $-v < h < w$ on U, i.e.,

$$g - (f_0 + \tfrac{1}{2}f_1) < h < g + (f_0 + \tfrac{1}{2}f_1) \quad \text{on } U. \qquad \square$$

Next we shall prove Proposition 2. Let

$$Q :=]0,1[\times]-1,1[^{d-2}\times]0,2[,\quad P :=]0,\infty[\times\mathbb{R}^{d-1},$$
$$S := \partial P = \{0\}\times\mathbb{R}^{d-1}.$$

Define $e_1 := (1,0,\dots,0)$, $e_d := (0,\dots,0,1)$ and, for every $n\in\mathbb{N}\cup\{0\}$,

$$x_n := 2^{-n}e_d,\quad B_n := \{x\in\mathbb{R}^d : ||x-x_n|| < 2^{-(n+2)}\}.$$

Clearly the balls B_n, $n\geqslant 0$, are disjoint. Because of the regularity of the half-balls $P\cap B_n$, there exist $\alpha_n > 0$ such that

$$z_n := x_n + \alpha_n e_1 \in P\cap B_n \quad\text{and}\quad \varepsilon_{z_n}^{(P\cap B_n)^c}((S\cap B_n)^c) < \frac{1}{4n}\quad (n\in\mathbb{N}). \tag{6}$$

Since line segments are polar, there exist strictly increasing continuous functions $s_n : [\alpha_n, 1]\to[0, 2^{-(n+2)}]$, $n\in\mathbb{N}$, such that $s_n(\alpha_n) = 0$ and the compact sets

$$L_n := \left\{ y\in\mathbb{R}^d : \alpha_n\leqslant y_1\leqslant 1, \sqrt{\sum_{i=2}^{d-1} y_i^2 + (y_d - 2^{-n})^2} \leqslant s_n(y_1)\right\}$$

satisfy

$$\varepsilon_{z_n}^{L_n}(\mathbb{R}^d) < \frac{1}{4n}. \tag{7}$$

By our construction, the sets L_n, $n\in\mathbb{N}$, are disjoint. Let T denote reflection at S, i.e., $T(y_1, y_2, \dots, y_d) = (-y_1, y_2, \dots, y_d)$ for $y = (y_1,\dots,y_d)\in\mathbb{R}^d$, and define

$$V := Q\setminus\bigcup_{n=1}^{\infty} L_n,\quad U := V\cup T(V)\cup B_0,\quad Z := \bigcup_{n=1}^{\infty}\{z_n, T(z_n)\}.$$

The set U is a simply connected domain such that all points $z\in\partial U\setminus Z$ are regular.

From now on we fix $n\in\mathbb{N}$. We choose $\varphi\in\mathscr{C}(\partial U)$ such that $0\leqslant\varphi\leqslant 1$ on $P\cap\partial U$, $\varphi = 1$ on $B_n\cap\partial L_n$, the support of φ is contained in $L_n\cup T(L_n)$, and $\varphi\circ T = -\varphi$. It is immediately seen that $0\leqslant H_U\varphi\leqslant 1$ on $P\cap U$ and

$$(H_U\varphi)\circ T = -H_U\varphi. \tag{8}$$

In particular, $\operatorname{osc}(H_U\varphi)\leqslant 1$, since all points in $S\cap\partial U$ are regular.

To prove that U has the desired property it therefore suffices to show that there is no function $h\in H(U)$ such that

$$|h - H_U\varphi|\leqslant 1 - \frac{1}{n}.$$

Suppose on the contrary that such a function h exists. Then $\tilde{h} := \frac{1}{2}(h - h\circ T)\in H(U)$, $\tilde{h}\circ T = -\tilde{h}$, and, by (8),

$$\tilde{h} - H_U\varphi = \tfrac{1}{2}(h - H_U\varphi) - \tfrac{1}{2}(h - H_U\varphi)\circ T$$

whence $|\tilde{h} - H_U\varphi|\leqslant 1 - 1/n$ as well. So we may assume that $h\circ T = -h$. In particular, $h = 0$ on $S\cap\bar{U}$ and of course $|h|\leqslant|H_U\varphi| + 1\leqslant 2$. Therefore

$$|h(z_n)| = |\varepsilon_{z_n}^{U^c}(h)|\leqslant 2\varepsilon_{z_n}^{U^c}((S\cap B_n)^c).$$

Let $V_n := (P \cap B_n) \setminus L_n$. By Bliedtner and Hansen [1, Proposition VI.9.4], $\varepsilon_{z_n}^{U^c}(S \cap B_n) \geqslant \varepsilon_{z_n}^{V_n^c}(S \cap B_n)$. Therefore

$$\varepsilon_{z_n}^{U^c}((S \cap B_n)^c) \leqslant \varepsilon_{z_n}^{V_n^c}((S \cap B_n)^c) \leqslant \varepsilon_{z_n}^{(P \cap B_n)^c}((S \cap B_n)^c) + \varepsilon_{z_n}^{L_n}((S \cap B_n)^c) < \frac{1}{2n},$$

where the last inequality follows from (6) and (7) (for the second inequality see [1, Proposition VI.9.3]). Thus $|h(z_n)| < 1/n$.

On the other hand, $\limsup_{x \to z_n} H_U \varphi(x) = 1$. Indeed, there is a sequence (y_m) of regular points in ∂U converging to z_n, and taking $x_m \in U$ such that $\|x_m - y_m\| < 1/m$ and $|H_U \varphi(x_m) - \varphi(y_m)| < 1/m$ we have $\lim_{m \to \infty} x_m = z_n$ and $\lim_{m \to \infty} H_U \varphi(x_m) = \lim_{m \to \infty} \varphi(y_m) = 1$. Since $h(z_n) = \lim_{x \in U, x \to z_n} h(x)$, we finally conclude that

$$\limsup_{x \in U, x \to z_n} |h - H_U \varphi|(x) > 1 - \frac{1}{n}.$$

This finishes the proof of Proposition 2.

We shall now combine Lemmas 6 and 7 for a proof of Theorem 3. So fix $g \in \mathscr{H}_b(U)$, $h_0 \in H(U)$ and $f \in \mathscr{C}(\bar{U})$ such that $\operatorname{osc}(g) < h_0$, $0 \leqslant f \leqslant h_0$ and $f = h_0$ on ∂U. There exists $\varepsilon > 0$ such that $\operatorname{osc}(g) < (1 - 2\varepsilon) h_0$. By Lemma 6, there exists $\varphi \in \mathscr{C}(\partial U)$ such that

$$|H_U \varphi - g| \leqslant \frac{1 - 2\varepsilon}{2} H_U h_0 = \frac{1 - 2\varepsilon}{2} h_0 \quad \text{on } U.$$

Taking $f_1 := (1 - 2\varepsilon) h_0$ and $f_0 := f + \varepsilon h_0$ we have $|H_U \varphi - g| \leqslant \frac{1}{2} f_1$ on U and $\operatorname{osc}(g) < f_0$ on $\bar{U}$ (recall that $\operatorname{osc}(g) = 0$ on U). So by Lemma 7, there exists $h \in H(U)$ such that

$$|g - h| < f_0 + \tfrac{1}{2} f_1 = \tfrac{1}{2} h_0 + f \quad \text{on } U$$

finishing the proof.

To show that the result is sharp let us look once more at the set U constructed in Proposition 2. Fix $n \in \mathbb{N}$. Using the same notation as before we define a function g on U by

$$g(x) := \begin{cases} H_V(\frac{3}{2}\varphi + \frac{1}{2} 1_{S \cap B_n})(x), & x \in V, \\ -H_V(\frac{3}{2}\varphi + \frac{1}{2} 1_{S \cap B_n})(Tx), & x \in T(V), \\ 0, & x \in S \cap B_0. \end{cases}$$

Then $|g| \leqslant 3/2$, $g \circ T = -g$, $g \in \mathscr{H}_b(U)$, and $\operatorname{osc}(g)(z) = 0$ for all points $z \in \partial U \setminus (Z \cup (S \cap \bar{B}_n))$.

We next show that $\operatorname{osc}(g) < 1 + 1/(4n)$. Indeed, obviously $\operatorname{osc}(g) = 1$ on $S \cap \bar{B}_n$. Furthermore, by definition of g and the minimum principle, we obtain that, for every

$x \in V_n = (P \cap B_n) \backslash L_n$,

$$\tfrac{1}{2} \leqslant g(x) + \varepsilon_x^{(P \cap B_n)^c}((S \cap B_n)^c),$$

where

$$\lim_{x \in U, x \to z_n} \varepsilon_x^{(P \cap B_n)^c}((S \cap B_n)^c) = \varepsilon_{z_n}^{(P \cap B_n)^c}((S \cap B_n)^c) < \frac{1}{4n}.$$

Therefore $\liminf_{x \to z_n} g(x) > 1/2 - 1/(4n)$. Since $0 \leqslant g \leqslant 3/2$ on V, we conclude that

$$\operatorname{osc}(g)(z_n) < 1 + \frac{1}{4n}.$$

By symmetry, $\operatorname{osc}(g)(Tz_n) < 1 + 1/(4n)$ as well. Finally, consider $m \in \mathbb{N}$, $m \neq n$. By minimum principle, for every $x \in (P \cap B_m) \backslash L_m$,

$$0 \leqslant g(x) \leqslant \tfrac{3}{2}\, \varepsilon_x^{(P \cap B_m)^c}((S \cap B_m)^c),$$

where

$$\lim_{x \in U, x \to z_m} \varepsilon_x^{(P \cap B_m)^c}((S \cap B_m)^c) = \varepsilon_{z_m}^{(P \cap B_m)^c}((S \cap B_m)^c) < \frac{1}{4m} \leqslant \frac{1}{4}.$$

So $\operatorname{osc}(g)(Tz_m) = \operatorname{osc}(g)(z_m) < 3/8$. Thus

$$\operatorname{osc}(g) < 1 + \frac{1}{4n}.$$

Take $f := h_0 := 1 + 1/(4n)$. To prove the sharpness of our result it suffices to show that there is no function $h \in H(U)$ such that

$$|g - h| \leqslant \frac{3}{2}\left(1 - \frac{1}{n}\right) \quad \text{on } U.$$

Indeed, suppose that we have such a function h. As in the proof of Proposition 2 we may assume that $h \circ T = -h$, $h = 0$ on $S \cap \bar{U}$ so that now $|h| \leqslant 3/2 + 3/2 = 3$, $|h(z_n)| < 3/(2n)$. Moreover,

$$\limsup_{x \to z_n} g(x) \geqslant \frac{3}{2} \limsup_{x \to z_n} H_U \varphi(x) = \frac{3}{2}.$$

Consequently,

$$\limsup_{x \in U, x \to z_n} |g - h|(x) > \frac{3}{2} - \frac{3}{2n} = \frac{3}{2}\left(1 - \frac{1}{n}\right)$$

proving our claim.

Instead of Proposition 4 which would be sufficient to prove Corollary 5 we present a more precise version:

Proposition 8. *Let $g \in \mathscr{H}_b(U)$, $f \in \mathscr{C}(\bar{U})$, and $h_0 \in H(U)$ strictly positive such that $|g + f| \leqslant h_0$ on U. Then, for every $\delta > 0$, there exist $\tilde{g} \in \mathscr{H}_b(U)$, $\tilde{f} \in \mathscr{C}(\bar{U})$ such that $\tilde{g} + \tilde{f} = g + f$ and $|\tilde{g}| < (3 + \delta) h_0$.*

Proof. Since $f \in \mathscr{C}(\bar{U})$ and

$$-f - h_0 \leqslant g \leqslant -f + h_0 \quad \text{on } U,$$

we have

$$-f - h_0 \leqslant g_* \leqslant g^* \leqslant -f + h_0 \quad \text{on } \bar{U},$$

i.e.,

$$\operatorname{osc}(g) = g^* - g_* \leqslant 2h_0 < (2 + \tfrac{2}{3}\delta)h_0.$$

By Theorem 3, there exists $h \in H(U)$ such that

$$|g - h| < \tfrac{3}{2}\left(2 + \tfrac{2}{3}\delta\right)h_0 = (3 + \delta)h_0. \qquad \square$$

Proof of Corollary 5. Let $g_n \in \mathscr{H}_{\mathrm{b}}(U), f_n \in \mathscr{C}(\bar{U})|_U$ such that the sequence $(g_n + f_n)$ converges uniformly. In order to show that the limit belongs to $\mathscr{H}_{\mathrm{b}}(U) + \mathscr{C}(\bar{U})|_U$ we may assume that $g_1 + f_1 = 0$ and that (taking a subsequence)

$$|(g_{n+1} + f_{n+1}) - (g_n + f_n)| < 2^{-n}.$$

Fix $h_0 \in H(U)$ such that $h_0 \geqslant 1$. By Proposition 8, there exist $\tilde{g}_n \in \mathscr{H}_{\mathrm{b}}(U)$ and $\tilde{f}_n \in \mathscr{C}(\bar{U})|_U$ such that

$$\tilde{g}_n + \tilde{f}_n = (g_{n+1} - g_n) + (f_{n+1} - f_n) \quad \text{and} \quad |\tilde{g}_n| < 4 \cdot 2^{-n} h_0$$

for every $n \in \mathbb{N}$. Then of course $|\tilde{f}_n| \leqslant |\tilde{g}_n + \tilde{f}_n| + |\tilde{g}_n| \leqslant 5 \cdot 2^{-n} h_0$. Define

$$g := \sum_{n=1}^{\infty} \tilde{g}_n, \quad f := \sum_{n=1}^{\infty} \tilde{f}_n.$$

Then $g \in \mathscr{H}_{\mathrm{b}}(U), f \in \mathscr{C}(\bar{U})|_U$ and

$$\lim_{n \to \infty} (g_n + f_n) = \sum_{m=1}^{\infty} (\tilde{g}_m + \tilde{f}_m) = g + f. \qquad \square$$

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